MATH2060 Solution 4

February 2022

6.4 Q17

Applying Taylor's Theorem 6.4.1 with $n = 1$ at the point $x = c$ shows that for all $x \in I$,

$$f(x) = f(c) + f'(c)(x - c) + \frac{1}{2}f''(s)(x - c)^2$$

for some $s \in (c, x)$ depending on x. By assumption, $f''(s) \geq 0$, and so $f(x) \geq f(c) + f'(c)(x - c)$ for all $x \in I$. This proves the result, since the right-hand-side of the inequality is precisely the equation of the tangent line to the graph of f at $(c, f(c))$.

6.4 Q19

Since f is certainly continuous on the interval $I = [2, 2.2]$, $f(2) = -1 < 0$, and $f(2.2) = 1.248 > 0$, the intermediate value theorem implies that there exists $2 < r < 2.2$ such that $f(r) = 0$.

By the usual rules of differentiation, $f'(x) = 3x^2 - 2$ and $f''(x) = 6x$. Since $f''(x) \geq 0$ on I, $f'(x)$ is increasing on I by Theorem 6.2.7, and so $f'(x) \geq f'(2) = 10$ for $x \in I$. On the other hand, $0 < f''(x) \leq 6 \cdot 2.2 < 14$ for $x \in I$. Applying Newton's method to the twice differentiable function f, together with the bounds $\left|f'(x)\right| \geq 10$ and $\left|f''(x)\right| \leq 14$ on I, we see (from equation (7) of the proof of 6.4.7) that for all $x' \in I$, the number $x'' := x' - \frac{f(x')}{f'(x')}$ satisfies

$$\left|x'' - r\right| \leq K\left|x' - r\right|^2,$$

where $K = \frac{14}{2 \cdot 10} = 0.7$.

In order to apply this inequality to the iterates $x' = x_n$ and $x'' = x_{n+1}$, one needs to ensure that all x_n lie in I. This can be seen[1] by retracing the proof of Newton's method 6.4.7 in finding the interval I^* of convergence, and then verifying that $x_1 = 2 \in I^*$. Indeed, according to the proof, one can choose a sufficiently small $\delta < 1/K$ to define $I^* := [r - \delta, r + \delta]$ so that it is contained entirely in I. In the setting of this question, because $1/K$ is already larger than the size of the interval $I = [2, 2.2]$, if $r \leq 2.1$, then one may take $I^* = [2, 2r - 2]$, and if $r \geq 2.1$, then $I^* = [2r - 2.2, 2.2]$. But $f(2.1) = 0.061 > 0$ (while $f(2) < 0$), the intermediate value theorem applied to f on the interval $[2, 2.1]$ implies that

[1] Alternatively, one could use elementary means to show that for all $x' \in I = [2, 2.2]$, we also have $x'' \in I$ for the function in this question.

$2 < r < 2.1$. Therefore $I^* = [2, 2r-2]$ contains the initial point $x_1 = 2$ and the sequence (x_n) satisfies

$$|x_{n+1} - r| \leq 0.7|x_n - r|^2 \qquad \text{for all } n \in \mathbb{N},$$

according to the result of 6.4.7.

From this, it follows that the error $e_n = x_n - r$ satisfies $|0.7e_{n+1}| \leq |0.7e_n|^2 \leq |0.7e_1|^{(2^n)}$ for $n \in \mathbb{N}$. Putting $n = 3$ shows that $|e_4| \leq 0.7^7 e_1^8$. Since $x_1, r \in I$ implies that $|e_1| \leq 0.2$, we have $|e_4| \leq 0.7^7 \cdot 0.2^8 = 2.10827008 \cdot 10^{-7} < 5 \cdot 10^{-7}$ and so x_4 is accurate to within 6 decimal places.

7.1 Q2

Given a tagged partition $\dot{\mathcal{Q}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ with tags $t_i = x_{i-1}$ at the left endpoint of the subintervals, the Riemann sum of a function $f : [a, b] \to \mathbb{R}$ is by definition

$$S(f; \dot{\mathcal{Q}}) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$

Similarly, given a tagged partition $\dot{\mathcal{Q}}' = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ with tags $t_i = x_i$ at the right endpoint of the subintervals, the Riemann sum of a function $f : [a, b] \to \mathbb{R}$ is by definition

$$S(f; \dot{\mathcal{Q}}') = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}).$$

Therefore,

(a) $S(f; \dot{\mathcal{P}}_1) = 0^2 \cdot (1-0) + 1^2 \cdot (2-1) + 2^2 \cdot (4-2) = 9$,

(b) $S(f; \dot{\mathcal{P}}_1) = 1^2 \cdot (1-0) + 2^2 \cdot (2-1) + 4^2 \cdot (4-2) = 37$,

(c) $S(f; \dot{\mathcal{P}}_2) = 0^2 \cdot (2-0) + 2^2 \cdot (3-2) + 3^2 \cdot (4-3) = 13$,

(d) $S(f; \dot{\mathcal{P}}_2) = 2^2 \cdot (2-0) + 3^2 \cdot (3-2) + 4^2 \cdot (4-3) = 33$.

7.1 Q7

We prove by induction that for any $f_1, \ldots, f_n \in \mathcal{R}[a, b]$ and $k_1, \ldots, k_n \in \mathbb{R}$, we have $f = \sum_{i=1}^n k_i f_i \in \mathcal{R}[a, b]$ and $\int_a^b f = \sum_{i=1}^n k_i \int_a^b f_i$. By Theorem 7.1.5(a), if $f_1 \in \mathcal{R}[a, b]$ and $k_1 \in \mathbb{R}$, then $k_1 f_1 \in \mathcal{R}[a, b]$ and $\int_a^b k_1 f_1 = k_1 \int_a^b f_1$. So the statement holds for $n = 1$. Suppose that the statement holds for $n = r$. Let $f_1, \ldots, f_{r+1}$ be functions in $\mathcal{R}[a, b]$ and $k_1, \ldots, k_{r+1} \in \mathbb{R}$. Then

$$f = \sum_{i=1}^{r+1} k_i f_i = \left(\sum_{i=1}^{r} k_i f_i \right) + k_{r+1} f_{r+1}.$$

By the induction hypothesis, $\sum_{i=1}^r k_i f_i \in \mathcal{R}[a, b]$; and by Theorem 7.1.5(a) $k_{r+1} f_{r+1} \in \mathcal{R}[a, b]$. Therefore, f, being the sum of these two functions, is in

$\mathcal{R}[a,b]$ by Theorem 7.1.5(b). Furthermore,

$$\begin{aligned}
\int_a^b f &= \int_a^b \left[\left(\sum_{i=1}^r k_i f_i\right) + k_{r+1} f_{r+1}\right] \\
&= \int_a^b \left(\sum_{i=1}^r k_i f_i\right) + \int_a^b k_{r+1} f_{r+1} \quad \text{(7.1.5b)} \\
&= \sum_{i=1}^r k_i \int_a^b f_i + k_{r+1} \int_a^b f_{r+1} \qquad \text{(induction hypothesis and 7.1.5a)} \\
&= \sum_{i=1}^{r+1} k_i \int_a^b f_i.
\end{aligned}$$

So the statement holds for $n = r + 1$. Thus the result follows by induction.

7.1 Q13

The proof of this question is similar to the one in Example 7.1.4b. Fix any $\epsilon > 0$ and choose $\delta = \epsilon/4\alpha > 0$. Let $\dot{\mathcal{P}}$ be any tagged partition of $[a,b]$ with norm $< \delta$. Define $\dot{\mathcal{P}}_1$ to be the subset of $\dot{\mathcal{P}}$ having its tags in $[a,c) \cup (d,b]$, where $\varphi(x) = 0$; and $\dot{\mathcal{P}}_2$ be the subset of $\dot{\mathcal{P}}$ having its tags in $[c,d]$, where $\varphi(x) = \alpha > 0$. Clearly, $S(\varphi; \dot{\mathcal{P}}_1) = 0$ and

$$S(\varphi; \dot{\mathcal{P}}) = S(\varphi; \dot{\mathcal{P}}_1) + S(\varphi; \dot{\mathcal{P}}_2) = S(\varphi; \dot{\mathcal{P}}_2).$$

Let U denote the union of subintervals in $\dot{\mathcal{P}}_2$. We claim that

$$[c + \delta, d - \delta] \subset U \subset [c - \delta, d + \delta].$$

To prove the first inclusion, take $u \in [c + \delta, d - \delta]$, which by assumption lies in an interval $I_k := [x_{k-1}, x_k]$ of $\dot{\mathcal{P}}$, and since $||\dot{\mathcal{P}}|| < \delta$, we have $x_k - x_{k-1} < \delta$. Then $x_{k-1} \le u \le d - \delta$ and $x_k \ge u \ge c + \delta$. So $x_k < x_{k-1} + \delta \le (d - \delta) + \delta = d$ and $x_{k-1} > x_k - \delta \ge c + \delta - \delta = c$. Thus the tag $t_k \in I_k$ must satisfy $c \le t_k \le d$, i.e. I_k is an interval of $\dot{\mathcal{P}}_2$ and $u \in I_k \subset U$.

The second inclusion is similar. Take $u \in U$, so that u lies in an interval $I_k := [x_{k-1}, x_k]$ of $\dot{\mathcal{P}}_2$, i.e. $c \le t_k \le d$. Since $x_k - x_{k-1} < \delta$, we have $x_{k-1} > x_k - \delta \ge t_k - \delta \ge c - \delta$, and $x_k < x_{k-1} + \delta \le t_k + \delta \le d + \delta$. Thus $u \in I_k \subset [c - \delta, d + \delta]$ because I_k is an interval.

U contains an interval of length $d - c - 2\delta$ and is contained in an interval of length $d - c + 2\delta$. Therfore

$$\alpha(d - c - 2\delta) \le S(\varphi; \dot{\mathcal{P}}_2) \le \alpha(d - c + 2\delta).$$

It follows that

$$\left|S(\varphi; \dot{\mathcal{P}}) - \alpha(d - c)\right| = \left|S(\varphi; \dot{\mathcal{P}}_2) - \alpha(d - c)\right| \le 2\alpha\delta < 4\alpha\delta = \epsilon.$$

Since $\dot{\mathcal{P}}$ is an arbitrary tagged partition with norm $< \delta$, this completes the proof.

7.1 Q14

(a) Since $0 \le x_{i-1} < x_i$,

$$x_{i-1}^2 = \frac{1}{3}(3x_{i-1}^2) \le \frac{1}{3}(x_{i-1}^2 + x_{i-1}x_i + x_i^2) = q_i^2 \le x_i^2.$$

Because q_i is the positive square root of the middle term in the inequalities, we have $x_{i-1} \le q_i \le x_i$.

(b) $Q(q_i)(x_i - x_{i-1}) = q_i^2(x_i - x_{i-1}) = \frac{1}{3}(x_{i-1}^2 + x_{i-1}x_i + x_i^2)(x_i - x_{i-1}) = \frac{1}{3}(x_i^3 - x_{i-1}^3)$.

(c) $S(Q;\dot{\mathcal{P}}) = \sum_{i=1}^n Q(q_i)(x_i - x_{i-1}) = \frac{1}{3}\sum_{i=1}^n (x_i^3 - x_{i-1}^3) = \frac{1}{3}(x_n^3 - x_0^3) = \frac{1}{3}(b^3 - a^3)$. The third equality holds because the sum telescopes.

(d) Fix any $\epsilon > 0$. Using uniform continuity of Q on $[a,b]$, we choose $\delta > 0$ such that for any $x, y \in [a,b]$ satisfying $|x-y| < \delta$, we have $\left|Q(x) - Q(y)\right| < \epsilon/(b-a)$. Let $\dot{\mathcal{P}} = \{(I_i, t_i)\}_{i=1}^n$ be any tagged partition of $[a,b]$ with norm $< \delta$, so that the endpoints of I_i satisfies $x_i - x_{i-1} < \delta$ for all $i = 1, \dots, n$. Let $\dot{\mathcal{Q}}$ have the same partition points, but with tags q_i as above. Since t_i and q_i both lie in I_i, we have $|t_i - q_i| < \delta$ and $\left|Q(t_i) - Q(q_i)\right| < \epsilon/(b-a)$. By part (c),

$$\begin{aligned}
\left|S(Q;\dot{\mathcal{P}}) - \frac{1}{3}(b^3 - a^3)\right| &= \left|S(Q;\dot{\mathcal{P}}) - S(Q;\dot{\mathcal{Q}})\right| \\
&= \left|\sum_{i=1}^n Q(t_i)(x_i - x_{i-1}) - \sum_{i=1}^n Q(q_i)(x_i - x_{i-1})\right| \\
&\le \sum_{i=1}^n \left|Q(t_i) - Q(q_i)\right| (x_i - x_{i-1}) \\
&< \frac{\epsilon}{b-a}\sum_{i=1}^n (x_i - x_{i-1}) = \frac{\epsilon}{b-a}(b-a) = \epsilon,
\end{aligned}$$

where we used the triangle inequality in the third step. This proves that $Q \in \mathcal{R}[a,b]$ and $\int_a^b Q = \frac{1}{3}(b^3 - a^3)$.